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Chapter 1
Gravitational Zero Point Energy and the Induced Cosmological Constant

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Abstract We discuss how to extract information about the cosmological constant from the Wheeler-DeWitt equation, considered as an eigenvalue of a Sturm-Liouville problem in a generic spherically symmetric background. The equation is approximated to one loop with the help of a variational approach with Gaussian trial wave functionals. A canonical decomposition of modes is used to separate transverse-traceless tensors (graviton) from ghosts and scalar. We show that no ghosts appear in the final evaluation of the cosmological constant. A zeta function regularization and a ultra violet cutoff are used to handle with divergences. A renormalization procedure is introduced to remove the infinities. We compare the result with the one obtained in the context of noncommutative geometries

1.1 Introduction

One of the biggest challenges of our century is the explanation of why the observed cosmological constant is so small when compared to the one estimated by Zero Point Energy (ZPE) computations in Quantum Field Theory. Indeed there exists a difference of 120 orders of magnitude between them. However, it appears that a definitive answer is still lacking. One possible approach to this problem comes from the Wheeler-DeWitt equation (WDW)[1], which is described by

$$\mathscr{H}\Psi = \left[(2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}\left({}^3R - 2\Lambda\right)\right]\Psi = 0, \tag{1.1}$$

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where $\kappa = 8\pi G$, G_{ijkl} is the super-metric and ${}^3R$ is the scalar curvature in three dimensions. The main reason to use such an equation is that its most general formulation intrinsically includes a cosmological term. Moreover, if we formally re-write the WDW equation as[1][2]

$$\frac{1}{V}\frac{\int \mathscr{D}\left[g_{ij}\right]\Psi^*\left[g_{ij}\right]\int_\Sigma d^3x\hat{\Lambda}_\Sigma\Psi\left[g_{ij}\right]}{\int \mathscr{D}\left[g_{ij}\right]\Psi^*\left[g_{ij}\right]\Psi\left[g_{ij}\right]} = \frac{1}{V}\frac{\left\langle\Psi\left|\int_\Sigma d^3x\hat{\Lambda}_\Sigma\right|\Psi\right\rangle}{\left\langle\Psi|\Psi\right\rangle} = -\frac{\Lambda}{\kappa}, \tag{1.2}$$

where

$$V = \int_\Sigma d^3x\sqrt{g} \tag{1.3}$$

is the volume of the hypersurface Σ and

$$\hat{\Lambda}_\Sigma = (2\kappa)\,G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{g}{}^3R/\left(2\kappa\right), \tag{1.4}$$

we recognize that the WDW equation can be represented by an expectation value. In particular, Eq.(1.2) represents the Sturm-Liouville problem associated with the cosmological constant. In this form the ratio Λ_c/κ represents the expectation value of $\hat{\Lambda}_\Sigma$ without matter fields. The related boundary conditions are dictated by the choice of the trial wave functionals which, in our case are of the Gaussian type. Different types of wave functionals correspond to different boundary conditions. The choice of a Gaussian wave functional is justified by the fact that we would like to explain the cosmological constant (Λ_c/κ) as a ZPE effect. To fix ideas, we will work with the following form of the metric

$$ds^2 = -N^2\left(r\right)dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1.5}$$

where $b\left(r\right)$ is subject to the only condition $b\left(r_t\right) = r_t$. As a first step, we begin to decompose the gravitational perturbation in such a way to obtain the graviton contribution enclosed in Eq.(1.2).

1.2 Extracting the graviton contribution

We can gain more information if we consider $g_{ij} = \bar{g}_{ij} + h_{ij}$,where $\bar{g}_{ij}$ is the background metric and h_{ij} is a quantum fluctuation around the background. Thus Eq.(1.2) can be expanded in terms of h_{ij}. Since the kinetic part of $\hat{\Lambda}_\Sigma$ is quadratic in the momenta, we only need to expand the three-scalar curvature $\int d^3x\sqrt{g}{}^3R$ up to the quadratic order. However, to proceed with the computation, we also need an orthogonal decomposition on the tangent space of 3-metric deformations[4, 5]:

$$h_{ij} = \frac{1}{3}\left(\sigma + 2\nabla\cdot\xi\right)g_{ij} + \left(L\xi\right)_{ij} + h_{ij}^{\perp}. \tag{1.6}$$

[1] See also Ref.[3] for an application of the method to a $f\left(R\right)$ theory.

The operator L maps ξ_i into symmetric tracefree tensors

$$(L\xi)_{ij} = \nabla_i \xi_j + \nabla_j \xi_i - \frac{2}{3} g_{ij} (\nabla \cdot \xi), \tag{1.7}$$

$h^{\perp}_{ij}$ is the traceless-transverse component of the perturbation (TT), namely $g^{ij} h^{\perp}_{ij} = 0$, $\nabla^i h^{\perp}_{ij} = 0$ and h is the trace of h_{ij}. It is immediate to recognize that the trace element $\sigma = h - 2(\nabla \cdot \xi)$ is gauge invariant. If we perform the same decomposition also on the momentum π^{ij}, up to second order Eq.(1.2) becomes

$$\frac{1}{V} \frac{\left\langle \Psi \left| \int_\Sigma d^3x \left[\hat{\Lambda}^{\perp}_\Sigma + \hat{\Lambda}^{\xi}_\Sigma + \hat{\Lambda}^{\sigma}_\Sigma \right]^{(2)} \right| \Psi \right\rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda}{\kappa}. \tag{1.8}$$

Concerning the measure appearing in Eq.(1.2), we have to note that the decomposition (1.6) induces the following transformation on the functional measure $\mathscr{D}h_{ij} \to \mathscr{D}h^{\perp}_{ij} \mathscr{D}\xi_i \mathscr{D}\sigma J_1$, where the Jacobian related to the gauge vector variable ξ_i is

$$J_1 = \left[\det \left(\triangle g^{ij} + \frac{1}{3} \nabla^i \nabla^j - R^{ij} \right) \right]^{\frac{1}{2}}. \tag{1.9}$$

This is nothing but the famous Faddev-Popov determinant. It becomes more transparent if ξ_a is further decomposed into a transverse part ξ^T_a with $\nabla^a \xi^T_a = 0$ and a longitudinal part $\xi^{\parallel}_a$ with $\xi^{\parallel}_a = \nabla_a \psi$, then J_1 can be expressed by an upper triangular matrix for certain backgrounds (e.g. Schwarzschild in three dimensions). It is immediate to recognize that for an Einstein space in any dimension, cross terms vanish and J_1 can be expressed by a block diagonal matrix. Since $\det AB = \det A \det B$, the functional measure $\mathscr{D}h_{ij}$ factorizes into

$$\mathscr{D}h_{ij} = \left(\det \triangle^T_V \right)^{\frac{1}{2}} \left(\det \left[\frac{2}{3} \triangle^2 + \nabla_i R^{ij} \nabla_j \right] \right)^{\frac{1}{2}} \mathscr{D}h^{\perp}_{ij} \mathscr{D}\xi^T \mathscr{D}\psi \tag{1.10}$$

with $\left(\triangle^{ij}_V \right)^T = \triangle g^{ij} - R^{ij}$ acting on transverse vectors, which is the Faddeev-Popov determinant. In writing the functional measure $\mathscr{D}h_{ij}$, we have here ignored the appearance of a multiplicative anomaly[6]. Thus the inner product can be written as

$$\int \mathscr{D}\rho \Psi^* \left[h^{\perp}_{ij} \right] \Psi^* \left[\xi^T \right] \Psi^* [\sigma] \Psi \left[h^{\perp}_{ij} \right] \Psi \left[\xi^T \right] \Psi [\sigma], \tag{1.11}$$

where

$$\mathscr{D}\rho = \mathscr{D}h^{\perp}_{ij} \mathscr{D}\xi^T \mathscr{D}\sigma \left(\det \triangle^T_V \right)^{\frac{1}{2}} \left(\det \left[\frac{2}{3} \triangle^2 + \nabla_i R^{ij} \nabla_j \right] \right)^{\frac{1}{2}}. \tag{1.12}$$

Nevertheless, since there is no interaction between ghost fields and the other components of the perturbation at this level of approximation, the Jacobian appearing in

the numerator and in the denominator simplify. The reason can be found in terms of connected and disconnected terms. The disconnected terms appear in the Faddeev-Popov determinant and these ones are not linked by the Gaussian integration. This means that disconnected terms in the numerator and the same ones appearing in the denominator cancel out. Therefore, Eq.(1.8) factorizes into three pieces. The piece containing $\hat{\Lambda}_{\Sigma}^{\perp}$ is the contribution of the transverse-traceless tensors (TT): essentially is the graviton contribution representing true physical degrees of freedom. Regarding the vector term $\hat{\Lambda}_{\Sigma}^{T}$, we observe that under the action of infinitesimal diffeomorphism generated by a vector field ε_i, the components of (1.6) transform as follows[4]

$$\xi_j \longrightarrow \xi_j + \varepsilon_j, \qquad h \longrightarrow h + 2\nabla\cdot\xi, \qquad h_{ij}^{\perp} \longrightarrow h_{ij}^{\perp}. \tag{1.13}$$

The Killing vectors satisfying the condition $\nabla_i\xi_j + \nabla_j\xi_i = 0$, do not change h_{ij}, and thus should be excluded from the gauge group. All other diffeomorphisms act on h_{ij} nontrivially. We need to fix the residual gauge freedom on the vector ξ_i. The simplest choice is $\xi_i = 0$. This new gauge fixing produces the same Faddeev-Popov determinant connected to the Jacobian J_1 and therefore will not contribute to the final value. We are left with

$$\frac{1}{V}\frac{\left\langle \Psi^{\perp}\left|\int_{\Sigma} d^3x \left[\hat{\Lambda}_{\Sigma}^{\perp}\right]^{(2)}\right|\Psi^{\perp}\right\rangle}{\left\langle \Psi^{\perp}|\Psi^{\perp}\right\rangle} + \frac{1}{V}\frac{\left\langle \Psi^{\sigma}\left|\int_{\Sigma} d^3x \left[\hat{\Lambda}_{\Sigma}^{\sigma}\right]^{(2)}\right|\Psi^{\sigma}\right\rangle}{\left\langle \Psi^{\sigma}|\Psi^{\sigma}\right\rangle} = -\frac{\Lambda^{\perp}}{\kappa} - \frac{\Lambda^{\sigma}}{\kappa}. \tag{1.14}$$

Note that in the expansion of $\int_{\Sigma} d^3x\sqrt{g}R$ to second order, a coupling term between the TT component and scalar one remains. However, the Gaussian integration does not allow such a mixing which has to be introduced with an appropriate wave functional. Extracting the TT tensor contribution from Eq.(1.2) approximated to second order in perturbation of the spatial part of the metric into a background term $\bar{g}_{ij}$, and a perturbation h_{ij}, we get

$$\hat{\Lambda}_{\Sigma}^{\perp} = \frac{1}{4V}\int_{\Sigma} d^3x\sqrt{\bar{g}}G^{ijkl}\left[(2\kappa)K^{-1\perp}(x,x)_{ijkl} + \frac{1}{(2\kappa)}\left(\tilde{\triangle}_L\right)_j^a K^{\perp}(x,x)_{iakl}\right], \tag{1.15}$$

where

$$\left(\tilde{\triangle}_L h^{\perp}\right)_{ij} = \left(\triangle_L h^{\perp}\right)_{ij} - 4R_i^k h_{kj}^{\perp} + {}^3R h_{ij}^{\perp} \tag{1.16}$$

is the modified Lichnerowicz operator and $\triangle_L$is the Lichnerowicz operator defined by

$$(\triangle_L h)_{ij} = \triangle h_{ij} - 2R_{ikjl}h^{kl} + R_{ik}h_j^k + R_{jk}h_i^k \qquad \triangle = -\nabla^a\nabla_a. \tag{1.17}$$

G^{ijkl} represents the inverse DeWitt metric and all indices run from one to three. Note that the term $-4R_i^k h_{kj}^{\perp} + {}^3R h_{ij}^{\perp}$ disappears in four dimensions. The propagator $K^{\perp}(x,x)_{iakl}$ can be represented as

$$K^{\perp}\left(\overrightarrow{x},\overrightarrow{y}\right)_{iakl}=\sum_{\tau}\frac{h_{ia}^{(\tau)\perp}\left(\overrightarrow{x}\right)h_{kl}^{(\tau)\perp}\left(\overrightarrow{y}\right)}{2\lambda\left(\tau\right)}, \tag{1.18}$$

where $h_{ia}^{(\tau)\perp}\left(\overrightarrow{x}\right)$ are the eigenfunctions of $\tilde{\triangle}_L$. τ denotes a complete set of indices and $\lambda\left(\tau\right)$ are a set of variational parameters to be determined by the minimization of Eq.$\left(1.15\right)$. The expectation value of $\hat{\Lambda}_{\Sigma}^{\perp}$ is easily obtained by inserting the form of the propagator into Eq.$\left(1.15\right)$ and minimizing with respect to the variational function $\lambda\left(\tau\right)$. Thus the total one loop energy density for TT tensors becomes

$$\frac{\Lambda}{8\pi G}=-\frac{1}{2}\sum_{\tau}\left[\sqrt{\omega_1^2\left(\tau\right)}+\sqrt{\omega_2^2\left(\tau\right)}\right]. \tag{1.19}$$

The above expression makes sense only for $\omega_i^2\left(\tau\right)>0$, where ω_i are the eigenvalues of $\tilde{\triangle}_L$. In the next section, we will explicitly evaluate Eq.$\left(1.19\right)$ for a background of spherically symmetric type.

1.3 One loop energy density

1.3.1 Conventional Regularization and Renormalization

The reference metric $\left(1.5\right)$ can be cast into the following form

$$ds^2=-N^2\left(r\left(x\right)\right)dt^2+dx^2+r^2\left(x\right)\left(d\theta^2+\sin^2\theta d\phi^2\right), \tag{1.20}$$

where

$$dx=\pm\frac{dr}{\sqrt{1-\frac{b(r)}{r}}} \tag{1.21}$$

and $b\left(r\right)$ a generic shape function. Specific examples are

$$b\left(r\right)=\frac{\Lambda_{dS}}{3}r^3;\qquad b\left(r\right)=-\frac{\Lambda_{AdS}}{3}r^3\qquad\text{and}\qquad b\left(r\right)=2MG. \tag{1.22}$$

However, we would like to maintain the form of the line element $\left(1.20\right)$ as general as possible. With the help of Regge and Wheeler representation[7], the Lichnerowicz operator $\left(\tilde{\triangle}_L h^{\perp}\right)_{ij}$ can be reduced to

$$\left[-\frac{d^2}{dx^2}+\frac{l\left(l+1\right)}{r^2}+m_i^2\left(r\right)\right]f_i\left(x\right)=\omega_{i,l}^2 f_i\left(x\right)\quad i=1,2\quad, \tag{1.23}$$

where we have used reduced fields of the form $f_i\left(x\right)=F_i\left(x\right)/r$ and where we have defined two r-dependent effective masses $m_1^2\left(r\right)$ and $m_2^2\left(r\right)$

$$\begin{cases} m_1^2(r) = \frac{6}{r^2}\left(1-\frac{b(r)}{r}\right) + \frac{3}{2r^2}b'(r) - \frac{3}{2r^3}b(r) \\ m_2^2(r) = \frac{6}{r^2}\left(1-\frac{b(r)}{r}\right) + \frac{1}{2r^2}b'(r) + \frac{3}{2r^3}b(r) \end{cases} \quad (r \equiv r(x)). \tag{1.24}$$

In order to use the W.K.B. method considered by ‘t Hooft in the brick wall problem[8], from Eq.(1.23) we can extract two r-dependent radial wave numbers

$$k_i^2\left(r,l,\omega_{i,nl}\right) = \omega_{i,nl}^2 - \frac{l(l+1)}{r^2} - m_i^2(r) \quad i=1,2 \quad . \tag{1.25}$$

Then the counting of the number of modes with frequency less than ω_i is given approximately by

$$\tilde{g}(\omega_i) = \int_0^{l_{\max}} \nu_i(l,\omega_i)(2l+1)\,dl. \tag{1.26}$$

$\nu_i(l,\omega_i)$ is the number of nodes in the mode with (l,ω_i), such that $(r \equiv r(x))$

$$\nu_i(l,\omega_i) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dx \sqrt{k_i^2(r,l,\omega_i)}. \tag{1.27}$$

Here it is understood that the integration with respect to x and $l_{\max}$ is taken over those values which satisfy $k_i^2(r,l,\omega_i) \geq 0$. With the help of Eqs.(1.26, 1.27), Eq.(1.19) becomes

$$\frac{\Lambda}{8\pi G} = -\frac{1}{\pi}\sum_{i=1}^{2}\int_0^{+\infty}\omega_i \frac{d\tilde{g}(\omega_i)}{d\omega_i}d\omega_i. \tag{1.28}$$

This is the one loop graviton contribution to the induced cosmological constant. The explicit evaluation of Eq.(1.28) gives

$$\frac{\Lambda}{8\pi G} = \rho_1 + \rho_2 = -\frac{1}{4\pi^2}\sum_{i=1}^{2}\int_{\sqrt{m_i^2(r)}}^{+\infty}\omega_i^2\sqrt{\omega_i^2 - m_i^2(r)}\,d\omega_i, \tag{1.29}$$

where we have included an additional 4π coming from the angular integration. The use of the zeta function regularization method to compute the energy densities ρ_1 and ρ_2 leads to

$$\rho_i(\varepsilon) = \frac{m_i^4(r)}{64\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{4\mu^2}{m_i^2(r)\sqrt{e}}\right)\right] \quad i=1,2 \quad , \tag{1.30}$$

where we have introduced the additional mass parameter μ in order to restore the correct dimension for the regularized quantities. Such an arbitrary mass scale emerges unavoidably in any regularization scheme. The renormalization is performed via the absorption of the divergent part into the re-definition of a bare classical quantity. Here we have two possible choices: the induced cosmological constant Λ or the gravitational Newton constant G. In any case a certain degree of arbitrariness is present because of the scale parameter μ. However, it is instructive a comparison of the result in Eq.(1.30) with the one which can be obtained by imposing a

UV cutoff. A direct calculation leads to $(i = 1,2)$

$$\int_{\sqrt{m_i^2(r)}}^{+\infty} \omega_i^2 \sqrt{\omega_i^2 - m_i^2(r)}\, d\omega_i$$

$$\underset{x_i=\omega_i/\sqrt{m_i^2(r)}}{=} \frac{m_i^4(r)}{4}\left[x_i^3\sqrt{x_i^2-1} - \frac{x_i}{2}\sqrt{x_i^2-1} - \frac{1}{2}\ln\left(x_i + \sqrt{x_i^2-1}\right)\right]_1^{\omega_{UV}/\sqrt{m_i^2(r)}}$$

$$\simeq \frac{m_i^4(r)}{4}\left[\frac{\omega_{UV}^4}{m_i^4(r)} - \frac{\omega_{UV}^2}{2m_i^2(r)} - \frac{1}{2}\ln\left(\frac{2\omega_{UV}}{\sqrt{m_i^2(r)}}\right)\right], \tag{1.31}$$

where $\omega_{UV} \gg \sqrt{m_i^2(r)}$. Nevertheless, for some backgrounds in some ranges,

$$m_0^2(r) = m_1^2(r) = -m_2^2(r)\,. \tag{1.32}$$

Thus, in these cases

$$\frac{\Lambda}{8\pi G} = \rho_1 + \rho_2 = -\frac{1}{4\pi^2}\left[\int_{\sqrt{m_0^2(r)}}^{+\infty} \omega^2\sqrt{\omega^2 - m_0^2(r)}\, d\omega + \int_0^{+\infty} \omega^2\sqrt{\omega^2 + m_0^2(r)}\, d\omega\right]$$

$$\simeq -\frac{1}{4\pi^2}\left[\frac{\omega_{UV}^4}{2} + \frac{m_0^4(r)}{8}\ln\left(\frac{m_0^2(r)\sqrt{e}}{4\omega_{UV}^2}\right)\right], \tag{1.33}$$

where we have used

$$\int_0^{+\infty} \omega^2\sqrt{\omega^2 + m_0^2(r)}\, d\omega$$

$$\underset{x=\omega/\sqrt{m_0^2(r)}}{=} \frac{m_0^4(r)}{4}\left[x^3\sqrt{x^2+1} + \frac{x}{2}\sqrt{x^2+1} - \frac{1}{2}\ln\left(x + \sqrt{x^2+1}\right)\right]_0^{\omega_{UV}/\sqrt{m_0^2(r)}}. \tag{1.34}$$

The Schwarzschild Schwarzschild-de Sitter (SdS) and Schwarzschild-Anti de Sitter (SAdS) backgrounds satisfy relation (1.32) in a region close to the throat. Indeed, by expanding $b(r)$ close to the throat, one gets $(r \equiv r(x))$

$$\begin{cases} m_1^2(r) = \frac{6}{r^2} - \frac{15r_t}{2r^3} - \frac{6b'(r_t)}{r^2} + \frac{15b'(r_t)r_t}{2r^3} \\ \\ m_2^2(r) = \frac{6}{r^2} - \frac{9r_t}{2r^3} - \frac{4b'(r_t)}{r^2} + \frac{9b'(r_t)r_t}{2r^3} \end{cases} \tag{1.35}$$

and for example, for the Schwarzschild case where $b(r) = r_t = 2MG$, we get

$$\begin{cases} m_1^2(r) = -\frac{3r_t}{2r^3} \\ \\ m_2^2(r) = +\frac{3r_t}{2r^3} \end{cases}. \tag{1.36}$$

Note that Eq.(1.39) works when the effective masses satisfy relation (1.32), otherwise the zeta function and the cutoff regularizations produce different results as shown by Eq.(1.31). The divergence can be eliminated by separating the cosmological constant Λ, into a bare cosmological constant Λ_0 and a divergent quantity Λ^{div}, where

$$\Lambda^{div} = \frac{Gm_0^4(r)}{\varepsilon 32\pi^2}, \tag{1.37}$$

or

$$\Lambda_{UV}^{div} = -\frac{G}{4\pi^2}\left[\frac{\omega_{UV}^4}{2} + \frac{m_0^4(r)}{8}\ln\left(\frac{\mu^2\sqrt{e}}{4\omega_{UV}^2}\right)\right]. \tag{1.38}$$

In both cases, the remaining finite value for the cosmological constant reads

$$\frac{\Lambda_0}{8\pi G} = (\rho_1(\mu) + \rho_2(\mu)) = \rho_{eff}^{TT}(\mu, r) = \frac{m_0^4(r)}{32\pi^2}\ln\left(\frac{4\mu^2}{m_0^2(r)\sqrt{e}}\right). \tag{1.39}$$

1.3.2 The example of Non Commutative theories

Non Commutative theories provide a powerful method to naturally regularize divergent integrals appearing in Eq.(1.29). Basically, the number of states is modified in the following way[11]

$$dn = \frac{d^3x d^3k}{(2\pi)^3} \Longrightarrow dn_i = \frac{d^3x d^3k}{(2\pi)^3}\exp\left(-\frac{\theta}{4}k_i^2\right), \tag{1.40}$$

with

$$k_i^2 = \omega_{i,nl}^2 - m_i^2(r) \quad i = 1, 2. \tag{1.41}$$

This deformation corresponds to an effective cut off on the background geometry (1.20). The UV cut off is triggered only by higher momenta modes $\gtrsim 1/\sqrt{\theta}$ which propagate over the background geometry. The virtue of this kind of deformation is its exponential damping profile, which encodes an intrinsic nonlocal character into fields $f_i(x)$. Plugging (1.27) into (1.26) and taking account of (1.40), the number of modes with frequency less than ω_i, $i = 1, 2$ is given by

$$\tilde{g}(\omega_i) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dx \int_0^{l_{\max}} (2l+1)\sqrt{\omega_{i,nl}^2 - \frac{l(l+1)}{r^2} - m_i^2(r)}\exp\left(-\frac{\theta}{4}k_i^2\right) dl \tag{1.42}$$

and the induced cosmological constant becomes

$$\frac{\Lambda}{8\pi G} = \frac{1}{6\pi^2}\left[\int_{\sqrt{m_0^2(r)}}^{+\infty}\sqrt{\left(\omega^2 - m_0^2(r)\right)^3}e^{-\frac{\theta}{4}\left(\omega^2 - m_0^2(r)\right)} + \int_0^{+\infty}\sqrt{\left(\omega^2 + m_0^2(r)\right)^3}e^{-\frac{\theta}{4}\left(\omega^2 + m_0^2(r)\right)}\right], \tag{1.43}$$

which integrated leads to

$$\frac{\Lambda}{8\pi G}=\frac{1}{12\pi^2}\left(\frac{4}{\theta}\right)^2\left(y\cosh\left(\frac{y}{2}\right)-y^2\sinh\left(\frac{y}{2}\right)\right)\ K_1\left(\frac{y}{2}\right)+y^2\cosh\left(\frac{y}{2}\right)K_0\left(\frac{y}{2}\right), \tag{1.44}$$

where $K_0(y)$ and $K_1(y)$ are the modified Bessel function and

$$y=\frac{m_0^2(r)\,\theta}{4}. \tag{1.45}$$

The asymptotic properties of (1.44) show that the one loop contribution is everywhere regular. Indeed, we find that when $y\to+\infty$,

$$\frac{\Lambda}{8\pi G}\simeq\frac{1}{6\pi^2\theta^2}\sqrt{\frac{\pi}{y}}\left[3+\left(8y^2+6y+3\right)\exp(-y)\right]\to 0. \tag{1.46}$$

Conversely, when $y\to 0$, we obtain

$$\frac{\Lambda}{8\pi G}\simeq\frac{4}{3\pi^2\theta^2}\left[2-\left(\frac{7}{8}+\frac{3}{4}\ln\left(\frac{y}{4}\right)+\frac{3}{4}\gamma\right)y^2\right]\to\frac{8}{3\pi^2\theta^2} \tag{1.47}$$

a finite value for Λ. Note that expression (1.44) can be used when the background satisfies the relation (1.32). For the other cases, we find that the effective masses contribute in the same way at one loop. Thus (1.43) becomes

$$\frac{\Lambda}{8\pi G}=\frac{1}{6\pi^2}\left[\int_{\sqrt{m_1^2(r)}}^{+\infty}\sqrt{\left(\omega^2-m_1^2(r)\right)^3}e^{-\frac{\theta}{4}\left(\omega^2-m_1^2(r)\right)}+\int_{\sqrt{m_2^2(r)}}^{+\infty}\sqrt{\left(\omega^2-m_2^2(r)\right)^3}e^{-\frac{\theta}{4}\left(\omega^2-m_2^2(r)\right)}\right]. \tag{1.48}$$

For example, when

$$m_1^2(r)=m_2^2(r), \tag{1.49}$$

Eq.(1.48) reduces to

$$\frac{\Lambda}{8\pi G}=\frac{1}{6\pi^2}\left(\frac{4}{\theta}\right)^2\left(\frac{1}{2}y(1-y)K_1\left(\frac{y}{2}\right)+\frac{1}{2}y^2K_0\left(\frac{y}{2}\right)\right)\exp\left(\frac{y}{2}\right). \tag{1.50}$$

The asymptotic expansion of Eq.(1.50) leads to

$$\frac{\Lambda}{8\pi G}\simeq\frac{1}{6\pi^2}\left(\frac{4}{\theta}\right)^2\frac{3}{8}\sqrt{\frac{\pi}{y}}\to 0, \tag{1.51}$$

when $y\to\infty$. On the other hand, when $z\to 0$, one gets

$$\frac{\Lambda}{8\pi G}\simeq\frac{1}{6\pi^2}\left(\frac{4}{\theta}\right)^2\left[1-\frac{z}{2}+\left(-\frac{7}{16}-\frac{3}{8}\ln\left(\frac{z}{4}\right)-\frac{3}{8}\gamma\right)z^2\right]\to\frac{8}{3\pi^2\theta^2}, \tag{1.52}$$

i.e. a finite value of the cosmological term.

1.4 Summary and Conclusions

In this contribution, the effect of a ZPE on the cosmological constant has been investigated using two specific geometries such as dS and AdS metrics. The computation has been done by means of a variational procedure with a Gaussian Wave Functional which should be a good candidate for a ZPE calculation. We have found that only the graviton is relevant[9]. Actually, the appearance of a ghost contribution is connected with perturbations of the shift vectors[4]. In this work we have excluded such perturbations. As usual, in ZPE calculation we meet the problem of divergences which are regularized with zeta function techniquesor by introducting a UV cutoff. After regularization , we have adopted to remove divergences by absorbing them into the induced cosmological constant Λ. Another possibility of keeping under control divergences comes from a NCG induced minimal length. As a result we get a modified counting of graviton modes. This let us obtain everywhere regular values for the cosmological constant, independently of the chosen background, which nevertheless is of a spherically symmetric type. Although the result seems to be promising, we have to note that the evaluation is at the Planck scale.

References

1. B. S. DeWitt, *Phys. Rev.* **160**, 1113 (1967).
2. R.Garattini, *J. Phys. A* **39**, 6393 (2006); gr-qc/0510061. R. Garattini, *J.Phys.Conf.Ser.* **33**, 215 (2006); gr-qc/0510062.
3. S. Capozziello and R. Garattini, *Class.Quant.Grav.* **24**, 1627 (2007); gr-qc/0702075.
4. D.V. Vassilevich, *Int. J. Mod. Phys.* **A 8**, 1637 (1993).
5. M. Berger and D. Ebin, *J. Diff. Geom.* **3**, 379 (1969). J. W. York Jr., *J. Math. Phys.*, **14**, 4 (1973); *Ann. Inst. Henri Poincaré* **A 21**, 319 (1974). P. O. Mazur and E. Mottola, *Nucl. Phys.* **B 341**, 187 (1990).
6. E. Elizalde, L. Vanzo and S. Zerbini, *Commun.Math.Phys.* **194**, 613 (1998), hep-th/9701060; E. Elizalde, A. Filippi, L. Vanzo and S. Zerbini, *Phys. Rev.* **D 57**, 7430 (1998), hep-th/9710171.
7. T. Regge and J. A. Wheeler, *Phys. Rev.* **108**, 1063 (1957).
8. G. 't Hooft, *Nucl. Phys.* **B 256**, 727 (1985).
9. P.A. Griffin and D.A. Kosower, *Phys. Lett.* **B 233**, 295 (1989).
10. R. Garattini, *TSPU Vestnik* **44 N7**, 72 (2004); gr-qc/0409016.
11. R. Garattini and P. Nicolini, *A noncommutative approach to the cosmological constant problem.* e-Print: arXiv:1006.5418 [gr-qc].